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On Strongly α-Regular Near-Rings

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ABSTRACT

This paper introduces and studies the concept of strongly α-regular near-rings and investigates their structural properties. New notions of left and right strongly α-regular near-rings are defined and examined in relation to α-regularity. Various results are established connecting strong α-regularity with conditions when like simple, reduced and subdirect irreducible under which α-regular near-rings become strongly α-regular are obtained, and the equivalence between left and right strongly α-regularity is discussed under suitable constraints. The role of idempotent elements and the absence of zero divisors are also explored. Furthermore, examples are provided to illustrate the concepts and to show that certain converses do not hold in general. The results contribute to the development of generalized regularity conditions in near-ring theory and provide a framework for further investigations involving endomorphism-based identities..

Keywords: α-regularity, regular near rings, strongly α-regular near-rings, zero divisors, idempotent..

INTRODUCTION

The concept of regularity was first introduced by John von Neumann in ring theory, where a ring R is called regular if for every $a \in R$, there exists $x \in R$ such that $a = axa$. This notion was later extended to near-rings by Günter Pilz, leading to the study of regular near-rings and their structural properties. Further strengthening of this concept resulted in the introduction of strongly regular rings by Gordon Mason, where the condition is refined to $a = a^2x$. These ideas have since been adapted and generalized within near-ring theory, providing a framework to study deeper algebraic structures and their properties. Motivated by these developments, the present work introduces and studies the concept of strongly α-regular near-rings, along with their left and right versions. These notions strengthen α-regularity and allow for a deeper analysis of the interaction between regularity and other structural properties. The introduction of strong conditions also helps in identifying when weaker forms of regularity can be elevated to stronger ones under additional assumptions.

2.PRELIMINARIES

Definition 2.1 [10]

A ***right near ring*** is a non-empty set N together with two binary operations "$+$" and "$\cdot$" such that

(i) $(N,+)$ is a group (not necessarily abelian)

(ii) $(N,\ \cdot)$ is a semigroup

(iii) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$

..:

Definition 2.2 [10]
A subgroup M of the additive group $(N, +)$ is called an ***N-subgroup*** of N if $NM \subseteq M$.

Definition 2.3 [10]
A near ring N is said to be ***regular near ring*** if for each $a \in N$, there exists $x \in N$ such that $axa = a$.

Definition 2.4 [10]
A near-ring N is said to be ***zero-symmetric*** if $a0 = 0$ for all $a \in N$.

Definition 2.5 [9]
A mapping $\alpha: N \to N$ is called a ***near-ring endomorphism*** if for all $a, b \in N$, $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(ab) = \alpha(a)\alpha(b)$.

Definition 2.6 [2]
A non-empty subset $I$of N is called an ***ideal*** of N if:
1. $(I, +)$ is a normal subgroup of $(N, +)$,
2. $IN \subseteq I$,
3. $a(b + i) - ab \in I$ for all $a, b \in N$ and $i \in I$.

Definition 2.7 [6]
A near-ring N is said to be ***commutative*** if $ab = ba$ for all $a, b \in N$.

Definition 2.8 [6]
An element a in N is ***nilpotent*** if $a^k = 0$ for some positive integer k.

Definition 2.9 [3]
A near-ring N is said to be ***reduced*** if it has no nonzero nilpotent elements.

Notation 2.10 [13]
L denotes the set of all nonzero nilpotent elements. We write $\boldsymbol{L} = \{\boldsymbol{0}\}$ if a near ring has no nonzero nilpotent elements.

Definition 2.11 [2]
The ***centre*** of a near-ring N is $C(N) = \{x \in N \mid xa = ax \text{ for all } a \in N\}$.

Definition 2.12 [10]
A near-ring N is said to fulfill the ***Insertion of Factors Property (IFP)*** provided that $\forall\ a, b, n \in N$ such that $ab = 0$ implies $anb = 0$.

Definition 2.13 [4]
The near ring N is said to be ***Boolean*** if $x^2 = x$ for each $x \in N$.

Definition 2.14 [6]
An element $e \in N$ is called ***idempotent*** if $e^2 = e$. The set of all idempotents is denoted by E.

Definition 2.15 [13]
An ideal I of N is said to be ***completely semi prime ideal*** if for all $x \in N$, $x^2 \in I \Longrightarrow x \in I$.

Definition 2.16 [11]
A near ring N is said to be ***S - near ring*** (***S' - near ring***) according as $x \in Nx$ ($x \in xN$)for all $x \in N$.

Definition 2.17 [9]
Let N be a right near-ring and $\alpha: N \to N$ a near-ring endomorphism. Then N is called ***α-regular*** if for every $a \in N$, there exists $x \in N$ such that $a = a\alpha(x)a$.
Theorem 2.18 [9]
If N is α-regular, then $a\alpha(x)$ and $\alpha(x)a$ are idempotents.

Proposition 2.19 [7]
A near ring N has no non zero nilpotent elements if and only if $x^2 = 0$ implies $x = 0$ for all $x \in N$.

Definition: 2.20 [1]
A near ring N is called a $\boldsymbol{P_k}$ ***near ring*** ($\boldsymbol{P'_k}$ ***near ring***) if there exists a positive integer k such that $x^k N = xNx$ ($Nx^k = xNx$) for all $x \in N$.

Definition: 2.21 [12]
A near ring N is said to be ***simple*** if the only ideal I of N are $\{0\}$ or N.

Definition 2.22 [5]
A mapping $\phi: N \to M$ is called ***anti-homomorphism*** of near-ring if for all $a, b \in N$, $\phi(a+b) = \phi(b) + \phi(a)$ and $\phi(ab) = \phi(b)\phi(a)$.

3.Main Results

Definition 3.1
Let N be a right near-ring and $\alpha: N \to N$ a near-ring endomorphism. Then N is called **left (right) $\boldsymbol{\alpha}$-regular** if for every $a \in N$, there exists $x \in N$ such that $a = a\alpha(x)a = \alpha(x)a^2$ ($a = a\alpha(x)a = a^2\alpha(x)$).

Definition 3.2
Let N be a right near-ring and $\alpha: N \to N$ a near-ring endomorphism. Then N is called **left (right) strongly $\boldsymbol{\alpha}$-regular** if for every $a \in N$, there exists $x \in N$ such that $a = \alpha(x)a^2$ ($a = a^2\alpha(x)$).
Remark 3.3
Obviously, every right(left) α-regular near ring is right(left) strongly α-regular near ring.

Example 3.4
Let then near ring $(N, +, \cdot)$ and is defined by the Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$ and the multiplication $(N, \cdot)$ defined as follows

$\cdot$	***0***	***a***	***b***	***c***
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

From the above remark we infer that right(left) α-regular near ring is a right(left) strongly α-regular near ring, the same can be observed in this example.
Remark 3.5
A right(left) strongly α-regular near ring need not necessarily be right(left) α-regular near ring, but addition of certain properties can suffice the statement.

Example 3.6
Let then near ring $(N, +, \cdot)$ and is defined by the Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$ and the multiplication $(N, \cdot)$ defined as follows

$\cdot$	***0***	***a***	***b***	***c***
0	0	0	0	0
a	0	b	a	c
b	0	b	a	c
c	0	c	c	c

Here N is right strongly α-regular near ring. Since there does not exist an element $x \in N$ for $a, b \in N$ such that $a = axa$ and $b = bxb$, it is not right α-regular near ring.

Theorem 3.7
A strongly α-regular near ring is an α-regular near ring if it satisfies each of the following properties.
i) $E \subseteq C(N)$
ii) Commutativity
Proof
i) Consider N to be right strongly α-regular near ring, for all $a \in N$ there exists a $x \in N$ such that $a = a^2\alpha(x)$
We have $E(N) \subseteq C(N)$, $\alpha(x)a$ and $a\alpha(x)$ are idempotents of N

$a = \big(a\alpha(x)\big)a = (\alpha(x)a)a = \alpha(x)a^2$
ii) Suppose N is commutative, then $ab = ba$ for all $a, b \in N$
Now consider a right strongly α-regular near ring,
$a = \big(a\alpha(x)\big)a = (\alpha(x)a)a = \alpha(x)a^2$
Thus N is right α-regular near-ring.
Similarly, we can prove for left strongly α-regular near ring.

Theorem 3.8
If N is left or right strongly α-regular then N is reduced.
Proof
Let $a^2 = 0$
In case of right strongly α-regular near ring the proof is trivial.
Now for left strongly α-regular we have for all $a \in N$ there exists a $x \in N$ such that $a = \alpha(x)a^2$
Since $a^2 = 0 \Longrightarrow a = \alpha(x).0$
Then $0 = a^2 = (\alpha(x).0)a = \alpha(x)(0.a) = \alpha(x).0 = a$
Hence N is reduced.

Remark 3.9
The converse of the above theorem need not be true in general but with inclusion of α-regularity the result can be obtained. We establish this in the following theorem.

Theorem 3.10
Let N be α-regular near ring and reduced then N is left or right strongly α-regular.
Proof
Given that N is a α-regular near ring, then for every $a \in N$, there exists $x \in N$ such that $a = a\alpha(x)a$.
To prove N is right strongly α-regular
We consider, $\big(a - a^2\alpha(x)\big)^2 = \big(a - a^2\alpha(x)\big)a - \big(a - a^2\alpha(x)\big)a^2\alpha(x)$

$$\begin{aligned} &= (a^2 - a^2\alpha(x)a) - \big(a^3\alpha(x) - a^2\alpha(x)a^2\alpha(x)\big) \\ &= (a^2 - a(a\alpha(x)a) - (a^3\alpha(x) - a(a\alpha(x)a)a\alpha(x) \\ &= (a^2 - a^2) - \big(a^3\alpha(x) - a^3\alpha(x)\big) = 0 \end{aligned}$$

Since N is reduced $a - a^2\alpha(x) = 0$
$\Longrightarrow a = a^2\alpha(x)$ and hence N is right strongly α-regular.
To prove N is left strongly α-regular
Now consider, $(a - \alpha(x)a^2)^2 = a(a - \alpha(x)a^2) - \alpha(x)a^2(a - \alpha(x)a^2)$

$$\begin{aligned} &= (a^2 - a\alpha(x)a^2) - (\alpha(x)a^3 - \alpha(x)a^2\alpha(x)a^2) \\ &= (a^2 - (a\alpha(x)a)a - (\alpha(x)a^3 - \alpha(x)a(a\alpha(x)a)a \\ &= (a^2 - a^2) - (\alpha(x)a^3 - \alpha(x)a^3) = 0 \end{aligned}$$

Since N is reduced $a - \alpha(x)a^2 = 0$
$\Longrightarrow a = \alpha(x)a^2$ and hence N is left strongly α-regular.

Theorem 3.11
Let N be both left and right strongly α-regular near-ring then it is α-regular and reduced.
Proof
From theorem 3.3, we can say N is reduced
Now given that N is both left and right strongly α-regular, then for all $a \in N$ there exists a $x \in N$ such that $a = \alpha(x)a^2 = a^2\alpha(x)$
Consider $(a - a\alpha(x)a)^2 = a(a - a\alpha(x)a) - a\alpha(x)a(a - a\alpha(x)a)$

$$\begin{aligned} &= (a^2 - a^2\alpha(x)a) - (a\alpha(x)a^2 - a\alpha(x)a^2\alpha(x)a)) \\ &= \big(a^2 - \big(a^2\alpha(x)\big)a\big) - (a(\alpha(x)a^2) - a(\alpha(x)a^2)\,\alpha(x)a)) \\ &= (a^2 - a^2) - (a^2 - a^2\,\alpha(x)a) \\ &= (a^2 - a^2) - (a^2 - a^2) = 0 \end{aligned}$$

Since N is reduced, $a - a\alpha(x)a = 0$
$\Longrightarrow a = a\alpha(x)a$
Therefore N is α- regular.

Theorem 3.12
If N is left or right α-regular then N has IFP.
Proof
Let $a \in N$ and $a.a = a^2 = 0$
Since N is a left α-regular, for all $a \in N$ there exists a $x \in N$ such that $a = a\alpha(x)a = \alpha(x)a^2$
Also $a^2 = 0 \Longrightarrow a = \alpha(x).0$
Then $0 = a^2 = (\alpha(x).0)a = \alpha(x)(0.a) = \alpha(x).0 = a$
So, we have $a = \alpha(x)a^2 = a\alpha(x)a = 0$
For $n = \alpha(x) \in N$,we write $ana = 0$
Hence N has IFP.

Theorem 3.13
A left α-regular near ring with IFP is right strongly regular.
Proof
Given that N is left α-regular, then for all $a \in N$ there exists a $x \in N$ such that $a = a\alpha(x)a = \alpha(x)a^2$
Then $\alpha(x)aa = a\alpha(x)a$
$\Longrightarrow \alpha(x)aa - a\alpha(x)a = 0$
$\Longrightarrow \big(\alpha(x)a - a\alpha(x)\big)a = 0$
Since N has IFP, there exists $n = \alpha(x) \in N$ such that $\big(\alpha(x)a - a\alpha(x)\big)\alpha(x)a = 0$
$\Longrightarrow \alpha(x)a\alpha(x)a - a\alpha(x)\alpha(x)a = 0$
$\Longrightarrow \alpha(x)a\alpha(x)a - a\alpha(x)^2a = 0$
$\Longrightarrow \alpha(x)a\alpha(x)a = a\alpha(x)^2a$
$\Longrightarrow \alpha(x)a = \alpha(x)a\alpha(x)a = a\alpha(x)^2a$
$\Longrightarrow a\alpha(x)a = a\alpha(x)a\alpha(x)a = a^2\alpha(x)^2a$
$\Longrightarrow a = a\alpha(x)a = a^2\alpha(x)^2a$
$\Longrightarrow a = a^2(\alpha(x)^2a)$
$\Longrightarrow a = a^2b$ where $b = \alpha(x)^2a$
Thus N is a right strongly regular near ring.

Theorem 3.14
Let N be an α-regular near-ring with $E \subseteq C(N)$, then it is left or right strongly α-regular.
Proof
Since N is an α-regular near ring, for all $a \in N$ there exists a $x \in N$ such that $a = a\alpha(x)a$
We have $E(N) \subseteq C(N)$, $\alpha(x)a$ and $a\alpha(x)$ are idempotents of N
$a = \big(a\alpha(x)\big)a = (\alpha(x)a)a = \alpha(x)a^2$
Thus N is left strongly α-regular near-ring.
$a = a(\alpha(x)a) = a\big(a\alpha(x)\big) = a^2\alpha(x)$
Thus N is right strongly α-regular near-ring.

Theorem 3.15
Let N have no zero divisors for $0 \neq a \in N$ the following are equivalent:
a) N is right strongly α-regular
b) N is left strongly α-regular
Proof
(a) $\Longrightarrow$ ***(b)***
Let N be right strongly α-regular near ring (i.e.) for all $a \in N$ there exists a $x \in N$ such that $a = a^2\alpha(x)$.

$$\begin{aligned} \text{Consider } a^2(a - \alpha(x)a^2) &= a^3 - a^2\alpha(x)a^2 \\ &= a^3 - \big(a^2\alpha(x)\big)a^2 \\ &= a^3 - (a)a^2 \\ &= a^3 - a^3 = 0 \end{aligned}$$

Since $a^2 \neq 0$, $a - \alpha(x)a^2 = 0$
$\Longrightarrow a = \alpha(x)a^2$
Hence N is left strongly α-regular.
(b) $\Longrightarrow$ ***(a)***
Let N be left strongly α-regular near ring (i.e.) for all $a \in N$ there exists a $x \in N$ such that $a = \alpha(x)a^2$.

$$\begin{aligned} \text{Consider } \big(a - a^2\alpha(x)\big)a^2 &= a^3 - a^2\alpha(x)a^2 \\ &= a^3 - (a^2\alpha(x))a^2) \end{aligned}$$

$= a^3 - (a)a^2$
$= a^3 - a^3 = 0$

Since $a^2 \neq 0$, $a - a^2\alpha(x) = 0$
$\Longrightarrow a = a^2\alpha(x)$
Hence N is right strongly α-regular.

Theorem 3.16
Let N be left strongly α-regular near-ring then N is simple.
Proof
Let N be a left strongly α-regular near ring (i.e.) for all $a \in N$ there exists a $x \in N$ such that $a = \alpha(x)a^2$.
Let I be an ideal of N
If $I = 0$ then the proof is complete.
If $I \neq 0$
Let $\alpha(x)$ be a non-zero element in I and $a \in N$
Then $a = \alpha(x)a^2 \in IN \subseteq I$
Hence $N \subseteq I$ and therefore $N = I$
We say N is simple.

Corollary 3.17
Since every simple near-ring is subdirectly irreducible, by theorem 3.15 we can infer that a left strongly α-regular near ring is subdirectly irreducible.

Theorem 3.18
A Boolean near-ring N that is left (right) strongly α-regular is an S-near-ring (S'-near-ring).
Proof
Consider N to be left strongly α-regular near ring (i.e.) for all $a \in N$ there exists a $x \in N$ such that $a = \alpha(x)a^2$.
Since N is Boolean and since $\alpha: N \to N$ is a near-ring endomorphism
We can write $a = \alpha(x)a \in Na$
Hence $a \in Na$ and therefore N is an S-near-ring.
Similarly, the definition of right strongly α-regular near ring becomes $a = a\alpha(x) \in aN$
Hence it is an S'-near-ring.

Theorem 3.19
Every ideal of right strongly α-regular is completely semi prime.
Proof
Let N be right strongly α-regular near ring (i.e.) for all $a \in N$ there exists a $x \in N$ such that $a = a^2\alpha(x)$.
And let I be an ideal of N and $a^2 \in I$
Then $a = a^2\alpha(x) \in IN$
Since I is an ideal of N , $IN \subseteq I$
Hence $a \in I$ and so N is completely semi prime.

Theorem 3.20
Homomorphic image of left(right) strongly α-regular near ring is left(right) strongly α-regular near ring if there exist $\overline{\alpha} : M \to M$ such that $\overline{\alpha} = \alpha \circ f$
Proof
Let $f: N \to M$ be a homomorphism and let α be an endomorphism
Consider $f(a) = f(\alpha(x)a^2)$
$= f(\alpha(x))f(a^2)$ [Since f is a homomorphism]
$= \alpha \circ f(x)f(a)^2$
$= \overline{\alpha}(x)f(a)^2$
Since f is a homomorphism to M there exists $b \in M$ such that $b = f(a)$
Therefore $b = \overline{\alpha}(x)b^2$ where $\overline{\alpha}(x)$ is an endomorphism of M.
Thus, Homomorphic image of left(right) strongly α-regular near ring is left(right) strongly α-regular near ring.

Corollary 3.21
Let N be left(right) strongly α-regular near ring. If I is any ideal of N then N/I is also a left(right) strongly α-regular near ring only if there exist $\overline{\alpha} : M \to M$ such that $\overline{\alpha} = \alpha \circ f$

Proof
The proof is taken care by above theorem.

Corollary 3.22
Anti-homomorphic image of left strongly α-regular near ring is right strongly α-regular near ring under the assumption if there exist $\bar{\alpha} : M \to M$ such that $\bar{\alpha} = \alpha \circ f$ and vice-versa.

Theorem 3.23
If N is right (left) α-regular then N is P_2 (P_2') near ring.
Proof
By definition of right α-regular for every $a \in N$, there exists $x \in N$ such that $a = a\alpha(x)a = a^2\alpha(x)$.
Consider $a\alpha(x)a = a^2\alpha(x)$
Since $x \in N$ and α is an endomorphism of N, $\alpha(x) \in N$
$\Rightarrow a\alpha(x)a \in aNa$ and $a^2\alpha(x) \in a^2N$
$\Rightarrow aNa = a^2N$
Therefore N is P_2 near ring.
Similarly, we can prove for left α-regular respectively..

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